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A uniform approximation for the fidelity in chaotic systems

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Abstract

In quantum/wave systems with chaotic classical analogues, wavefunctions evolve in highly complex, yet deterministic ways. A slight perturbation of the system, though, will cause the evolution to diverge from its original behaviour increasingly with time. This divergence can be measured by the fidelity, which is defined as the squared overlap of the two time evolved states. For chaotic systems, two main decay regimes of either Gaussian or exponential behaviour have been identified depending on the strength of the perturbation. For perturbation strengths intermediate between the two regimes, the fidelity displays both forms of decay. By applying a complementary combination of random matrix and semiclassical theory, a uniform approximation can be derived that covers the full range of perturbation strengths. The time dependence is entirely fixed by the density of states and the so-called transition parameter, which can be related to the phase space volume of the system and the classical action diffusion constant, respectively. The accuracy of the approximations is illustrated with the standard map.

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1. Introduction

In our contribution to this special issue on random matrix theory (RMT), we address the subject of wave field evolution governed by linear wave equations. In particular, our interest is in the sensitivity of evolution to perturbations. The treatment of this problem illustrates two well-known, important lessons arising in RMT studies. First, RMT and semiclassical theories, when used in tandem, give a much stronger theoretical approach to a problem than either was capable of on its own [1]. Second, even though RMT is inherently unphysical in certain ways, nevertheless, careful application leads to physically meaningful predictions [2]. For example, using RMT to mock up the evolution of a wave packet generates a dispersal into the entire Hilbert space simultaneously. Thus, it lacks a dynamical continuity property that a semiclassical theory would easily reveal as absurd. It turns out that for a wave mechanical

system with chaotic underlying ray dynamics, there is a logarithmic timescale dependence on the wave vector [3–5], and RMT does not 'know' about this; i.e. its results cannot be trusted for wave propagation up to this timescale. Yet, beyond the so-called logtime (where the semiclassical dynamics increase in complexity exponentially [6]), RMT can be a powerful tool for understanding the statistical nature of the dynamics. In what follows, we take advantage of the RMT-semiclassical complementarity and avoid making inferences about pre-logtime dynamics.

The study of the sensitivity of wave field evolution to perturbation has recently attracted a great deal of interest [7–13] for several fundamental reasons, i.e. decoherence, reversibility, tomography, etc. In the case in which the basic system is weakly coupled to an environment, the sensitivity relates to the manner in which the system decoheres [14]. The recent focus on quantum computation has greatly increased the attention given to this issue [15]. Another motivation has been the reversibility of wave field evolutions such as the spin polarization echoes in nuclear magnetic resonance [16]. For a broad class of problems, propagating forward in time with two slightly different systems is mathematically equivalent to propagating forward with one system and reversing the dynamics with another for the same propagation time. Another application is to problems of waves propagating through random media (WPRM), the medium is likely partially characterized and often slowly evolving; twinkling of starlight passing through the atmosphere is an example. The interest is often in performing tomography or at least knowing what information can be deduced about the medium [17]. Similar statements can be made regarding the study of disordered systems [18]. Lastly, we mention that problems involving simple, chaotic systems or even strongly interacting, many-body systems usually require taking measurements as a function of some system parameter. For these systems, single measurements may almost seem like random number generation, whereas it turns out that important information can be extracted from the parameter variations of the measurements [19]. The trick is to know what information therein exists. Our focus here will remain on the wave field evolution behaviour without getting into specific physical applications.

Introduced nearly 20 years ago, a natural measure of the sensitivity of wave field evolution is the fidelity, which is defined as the squared overlap of some initial state propagated forward in time with two slightly different systems [20]. For normalized wavefunctions, it begins at unity and typically decays to some small value. In the spin polarization experiments previously cited, it was observed that a change in the functional form of the fidelity's decay occurred depending upon the dipolar interactions [21]. The decay exhibited either Gaussian or exponential behaviours and in some instances both. This mirrors known behaviour found in simple, quantized chaotic systems [8, 9]; note that there also exists some work on integrable, near-integrable and mixed phase-space systems [22–24] which exhibit other behaviours. For extremely weak perturbation strengths, quantum perturbation theory along with some simplifying assumptions quickly lead to the prediction of a Gaussian decay [20]. For larger perturbation strengths, this approach fails, and a Fermi golden rule regime emerges which implies an exponential decay behaviour [25]. Semiclassical theory gives this decay rate in terms of a classical action diffusion constant [9]. If the perturbation strength is even larger, at some point the fidelity decay rate saturates because it is impossible to decay faster than the logtime [7]; this is known as the Lyapunov regime because the rate is independent of the perturbation strength, and often determined by the Lyapunov exponent.

Regardless of the strength of the perturbation or the nature of the dynamics, for very short times the fidelity is quadratic. This regime which can be derived by using time-dependent perturbation theory may be extremely short in time and difficult to observe. Likewise, in the long time limit, the fidelity will saturate to the inverse of the effective number of states

in the system. Both of these effects decrease in importance in the asymptotic limit, and are disregarded in what follows; the Lyapunov regime is also disregarded since that involves pre-logtime dynamics.

In this paper, we concentrate on developing a uniform approximation that encompasses both the Gaussian and exponential decays as well as the intermediate regime for simple chaotic systems; and we restrict our attention to the Schrödinger wave equation. The fidelity undergoes a crossover from exponential to Gaussian behaviour as a function of time, and the value of the perturbation strength determines whether the crossover is visible. In fact, it turns out that the only information residing in the decay of the fidelity is the density of states and the transition parameter of RMT [26]; the spreading width could also substitute for the transition parameter. Alternatively, using semiclassical arguments, one could also say that the only information contained is the phase space volume and the classical action diffusion constant [9].

The paper is organized as follows. In the next section, we review the Gaussian and exponential Fermi golden rule regimes found in simple quantized chaotic systems. In section 3, a uniform approximation is derived using RMT and a degenerate perturbation theory. The theory is applied to the quantized standard map in section 4. It is found to give an excellent approximation of the quantum results. We then conclude with some discussion and closing remarks.

2. The fidelity

Consider an evolving quantum wavefunction $|\alpha_\lambda(t)\rangle$ where λ defines the parameters of a Hamiltonian, $H(\lambda)$. The overlap of the same initial state, $|\alpha\rangle$, propagated via two different Hamiltonians specified by λ_1 and λ_2 is

$$A_\lambda(\epsilon; t) = \langle \alpha_{\lambda_1}(t) | \alpha_{\lambda_2}(t) \rangle = \langle \alpha | \hat{U}^\dagger_{\lambda_1}(t) \hat{U}_{\lambda_2}(t) | \alpha \rangle \tag{1}$$

where $\hat{U}_\lambda(t)$ is the unitary evolution operator, $\epsilon = \lambda_1 - \lambda_2$ and $\lambda = (\lambda_2 + \lambda_1)/2$. In the limit that the system is strongly chaotic, the statistical properties of the overlap are independent of λ, so we omit the λ subscript from here on. The fidelity is just the absolute square overlap

$$\mathcal{C}(\epsilon; t) = |A(\epsilon; t)|^2. \tag{2}$$

It is sufficient to work with the overlap itself in developing the theory since the operation of squaring only introduces cross-correlation effects higher order in ϵ that can be ignored.

By inserting complete sets of states, the overlap can be rewritten in terms of the eigenenergies and eigenstates of the perturbed and unperturbed systems as

$$A(\epsilon; t) = \sum_n \sum_m \langle \alpha(0) | n_{\lambda_1} \rangle \langle n_{\lambda_1} | m_{\lambda_2} \rangle \langle m_{\lambda_2} | \alpha(0) \rangle \exp\{-\mathrm{i}[E_m(\lambda_2) - E_n(\lambda_1)]t/\hbar\}. \tag{3}$$

The Bohigas–Giannoni–Schmit conjecture [27] and later works suggest that for a chaotic system, it is appropriate to apply RMT arguments to understand its statistical behaviour. Here, as mentioned already, one has to add the caveat—as long as ϵ is weak enough that $\mathcal{C}(\epsilon; t)$ decays on a much longer timescale than the logtime. The first property we would like to invoke is the invariance of the ensemble over changes in bases. Thus, any rotation to a different initial state must generate the same decay behaviour for the fidelity. It suffices to make an average over a complete set of initial wavefunctions $|\alpha(0)\rangle$ to obtain

$$\overline{A(\epsilon; t)} = \frac{1}{N} \sum_n \sum_m |\langle n_{\lambda_1} | m_{\lambda_2} \rangle|^2 \exp\{-\mathrm{i}[E_m(\lambda_2) - E_n(\lambda_1)]t/\hbar\} \tag{4}$$

where N is the number of states in the complete set. RMT has been shown to be a strongly ergodic theory [2, 28], which in this case implies that the fluctuations of the quantity $\delta A = A(\epsilon; t) - \overline{A(\epsilon; t)}$ vanish with increasing matrix dimensionality as N^{-1}.

The averaged overlap may be decomposed into diagonal and off-diagonal parts

$$\overline{A(\epsilon; t)} = \frac{1}{N}\sum_n \left|\langle n_{\lambda_1}|n_{\lambda_2}\rangle\right|^2 \exp[-\mathrm{i}\Delta E_n t/\hbar] + \frac{1}{N}\sum_n \sum_{m\neq n} \left|\langle n_{\lambda_1}|m_{\lambda_2}\rangle\right|^2 \exp\{-\mathrm{i}[E_m(\lambda_2) - E_n(\lambda_1)]t/\hbar\} \tag{5}$$

where $\Delta E_n = E_n(\lambda_2) - E_n(\lambda_1)$ is the change in an eigenenergy due to the perturbation. For $\epsilon = 0$, the off-diagonal term exactly vanishes, the diagonal term equals unity and the fidelity does not decay as must be the case.

2.1. The quantum perturbative regime

The diagonal term is defined by adiabatically following an eigenlevel as the system parameters are continuously varied. It may be further evaluated by ensemble averaging

$$\langle\overline{A_{\mathrm{diag}}(\epsilon; t)}\rangle = \frac{1}{N}\sum_n \left\langle \left|\langle n_{\lambda_1}|n_{\lambda_2}\rangle\right|^2 \exp\left[\frac{-\mathrm{i}\Delta E_n t}{\hbar}\right]\right\rangle. \tag{6}$$

It is most significant for extremely small perturbation strengths. From quantum perturbation theory, the lowest order shift of an eigenvalue depends on the diagonal element of the perturbation whereas the rotation of the eigenstates depends on off-diagonal elements and energy differences. Within RMT the perturbation matrix elements are uncorrelated and thus to lowest order in ϵ, the ensemble averaging of the amplitudes and phases can be separated. The average over the phases is over a Gaussian probability density for the diagonal matrix elements. Also, the ensemble average of the amplitude is independent of n. Thus,

$$\langle\overline{A_{\mathrm{diag}}(\epsilon; t)}\rangle \approx \left\langle\left|\langle n_{\lambda_1}|n_{\lambda_2}\rangle\right|^2\right\rangle \exp\left(-\epsilon^2\sigma_v^2 t^2/2\hbar^2\right) \tag{7}$$

where σ_v^2 is the variance of the diagonal matrix elements of the perturbation V defined by writing the Hamiltonian locally as $H(\lambda) = H_0 + \epsilon V$. In the limit of a differential ϵ, σ_v^2 is equal to the variance of the level slopes by the Hellmann–Feynman theorem [29]. If the perturbations are sufficiently small such that the eigenfunctions do not significantly vary, i.e. $\langle n_{\lambda_1}|m_{\lambda_2}\rangle \approx \delta_{nm}$, then the off-diagonal terms will be negligible and may be ignored. Hence, the fidelity has a Gaussian decay in time in the weak perturbation regime.

The diagonal matrix element (level slope) variance is not a free parameter, and can be given an elegant interpretation through a semiclassical approach [30–32]. It is given by

$$\sigma_v^2 \approx \frac{2gK(E)}{\pi\hbar\overline{d}\beta} \tag{8}$$

where $2g/\beta$ is the number of classical orbits with identical action and $\overline{d}$ is the total mean density of states. The index $\beta = 1$ for time-reversal-invariant systems and $\beta = 2$ for time-reversal-breaking systems. $K(E)$ is the classical action diffusion constant on the energy surface E given by [33]

$$K(E) = \int_0^\infty \langle V(\mathbf{p}(0), \mathbf{q}(0); \lambda) V(\mathbf{p}(t), \mathbf{q}(t); \lambda)\rangle_{po}\, \mathrm{d}t. \tag{9}$$

In this expression the averaging is defined over the primitive periodic orbits of long period. The physical picture is that the action difference of two long orbits continuously deformable

into each other as a function of ϵ is the result of a diffusive process. As the orbit explores the phase space 'randomly', sometimes its action increases relative to the other orbit, sometimes it decreases. The variance of that diffusion is proportional to the variance of the level slopes.

2.2. Semiclassical analysis

The above analysis describes a very restricted range of perturbation ϵ centred at zero. A semiclassical approach without the quantum perturbation limitations can be developed that is valid over a much broader range. The semiclassical construction of an evolving wavefunction begins with the propagator

$$\langle \mathbf{q}|\hat{U}|\mathbf{q}'\rangle \approx \left(\frac{1}{2\pi \mathrm{i}\hbar}\right)^{d/2} \sum_j \left|\det\left(\frac{\partial^2 W_j(\mathbf{q},\mathbf{q}';t)}{\partial \mathbf{q}\partial \mathbf{q}'}\right)\right|^{1/2} \exp\left(\mathrm{i}W_j(\mathbf{q},\mathbf{q}';t)/\hbar - \frac{\mathrm{i}\pi\nu_j}{2}\right). \quad (10)$$

The phase is specified by the time integral of the Lagrangian $W_j(\mathbf{q},\mathbf{q}';t)$ and an index based on the properties of the conjugate points (such as focal points), ν_j.

The overlap decay in terms of the propagator is

$$A(\epsilon;t) = \int \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{q}''\langle\alpha|\mathbf{q}\rangle\langle\mathbf{q}|\hat{U}^{\dagger}_{\lambda_1}|\mathbf{q}'\rangle\langle\mathbf{q}'|\hat{U}_{\lambda_2}|\mathbf{q}''\rangle\langle\mathbf{q}''|\alpha\rangle. \quad (11)$$

For the reasons given in the discussion surrounding equation (4), there can be no overlap decay dependence on the specific initial state for chaotic systems as long as the decay is on a timescale which is much longer than the logtime. Since we have explicitly excluded the Lyapunov regime from consideration in this paper, we are free to choose any form for the initial state. For convenience, we take Gaussian wave packets

$$\langle \mathbf{q}|\alpha\rangle = (\pi\sigma^2)^{-d/4} \exp\left[-\frac{(\mathbf{q}-\mathbf{q}_\alpha)^2}{2\sigma^2} + \frac{\mathrm{i}\mathbf{p}_\alpha}{\hbar}(\mathbf{q}-\mathbf{q}_\alpha)\right]. \quad (12)$$

The initial coordinates of the actions are expanded about the centre of the wave packet

$$W_j(\mathbf{q},\mathbf{q}';t) = W_j(\mathbf{q},\mathbf{q}_\alpha;t) - \mathbf{p}_\alpha\cdot(\mathbf{q}'-\mathbf{q}_\alpha) + \frac{1}{2}(\mathbf{q}'-\mathbf{q}_\alpha)^T \left.\frac{\partial W_j(\mathbf{q},\mathbf{q}';t)}{\partial \mathbf{q}'\partial \mathbf{q}'}\right|_{\mathbf{q}_\alpha} (\mathbf{q}'-\mathbf{q}_\alpha). \quad (13)$$

Changes in the amplitudes are ignored, since the most important contribution to the overlap decay is each orbit's action change due to its division by $\hbar$. Applying stationary phase on the $\mathbf{q}$ and $\mathbf{q}''$ integrals, we obtain

$$A(\epsilon;t) \approx \int \sum_j c_j \exp\left[\frac{\mathrm{i}}{\hbar}\Delta W_j(\mathbf{q},\mathbf{q}_\alpha;t)\right] \mathrm{d}\mathbf{q} \quad (14)$$

where c_j is a magnitude which disappears from the expressions ahead, but could be deduced using the equations in the appendix of [31]. The $\Delta W_j(\mathbf{q},\mathbf{q}_\alpha;t)$ are the action differences of two rays that begin at $\mathbf{q}_\alpha$ and end at position $\mathbf{q}$ in time t and are continuously deformable into each other; bifurcations are neglected. First-order classical perturbation theory gives

$$\Delta W_j(\mathbf{q},\mathbf{q}_\alpha;t) = \epsilon \int_0^t \frac{\partial L_j(\mathbf{q},\mathbf{q}_\alpha;t')}{\partial\lambda}\,\mathrm{d}t'. \quad (15)$$

The stability matrix elements and the Lagrangian are evaluated along the orbits of $H(\lambda)$. In general, stationary phase integration cannot be performed on the last integral because the action differences are less than $\hbar$.

Again a statistical argument can be employed. The weighted phases will be nearly Gaussian distributed for the reason mentioned earlier that long orbit action changes are part of a diffusion process [9]. The last integral and summation in equation (14) can be replaced by a Gaussian integral. Thus, we obtain

$$A(\epsilon; t) \approx \exp\left(-\sigma_W^2 / 2\hbar^2\right) \tag{16}$$

where σ_W^2 is the variance of the action differences and is given by [30, 31]

$$\sigma_W^2 = 2\epsilon^2 K(E)t = \left(2\pi \overline{d}\sigma_v^2\right)\epsilon^2 \hbar\beta t/2g \tag{17}$$

with the same action diffusion constant as before. In contrast to the quantum perturbative argument giving a quadratic t-dependence in the exponential, here the argument is linear. The semiclassical argument does not contain the action correlations necessary for the theory to contain the quantization of the spectrum. It therefore misses the Gaussian contribution from the diagonal terms, and essentially evaluates the off-diagonal contribution in the regime of larger perturbations as though there is a continuum. The last form given in the equation exhibits the Fermi golden rule form expected for such circumstances. Note that the symmetry factors do not enter $K(E)$ (the symmetry dependences cancel in the final form) in contrast to the diagonal term.

3. A uniform approximation

A straightforward argument determines the value of ϵ which separates the exponential and Gaussian regimes. We emphasize that this is *not* the crossover to exponential dependence beginning from the quadratic time dependence derived using time-dependent perturbation theory, but rather the relative strength of contributions from diagonal versus off-diagonal contributions in equation (5). Note that either dependence dominates over the time range in which the other's argument of the exponential takes on the lesser value. A crossover between the two regimes occurs when their arguments are equal. Simple algebra gives the timescale $t^* = h\bar{d}\beta/2g$, which is the Heisenberg time, τ_H, to within the symmetry factor. Exponential decay dominates if the decay is completed by τ_H, and Gaussian decay dominates if little decay has occurred by τ_H. In between it turns out that the decay has components of both behaviours. In terms of the parameters, the question is whether

$$\epsilon^2 \overset{?}{\gtrless} \frac{\hbar^2}{K(E)t^*} = \frac{\hbar g}{\pi \bar{d}\beta K(E)}. \tag{18}$$

It turns out using the results derived next that near the equality, the magnitude of the off-diagonal term is comparable to the diagonal term.

Returning to equation (7), it is an oversimplification to take the amplitude equal to unity. Instead, we define the fraction

$$f = \left\langle \left| \langle n_{\lambda_1} | n_{\lambda_2} \rangle \right|^2 \right\rangle \tag{19}$$

and ahead give a theory for its value in the neighbourhood of unity (small ϵ) using a degenerate perturbation theory and RMT. It is worth noting the broader significance of the quantity f. It and some closely related measures have previously been studied by several authors. In [34], the authors give the leading analytic correction from unity for the case of broken time reversal symmetry, and infer the asymptotic behaviour for large ϵ, and in [35, 36], a rough guess is given for the full range of behaviour consistent with numerical simulations with random matrices. These measures' interpretations as parametric correlators of eigenstates, eigenstate components, or widths were emphasized. Motivations for their introduction come from a broad

variety of possibilities of parametric dependence on controllable external or uncontrollable variables such as electromagnetic fields, thermodynamics quantities, or shape/geometric properties. Physical realizations from nuclear fission to quantum dots to atomic and molecular spectroscopy are possible. It is interesting to note that the same quantity is entering into the theory of the fidelity, but only for transition values of ϵ between the two domains giving exponential and Gaussian decay.

The argument giving a Gaussian behaviour for the phase average remains valid to a regime where f is quite small and the diagonal term negligible. It suffices to evaluate f to complete a more broadly applicable expression for the diagonal term.

The off-diagonal term is more subtle. By normalization, the prefactor must be equal to $1 - f$, but the argument of the exponential needs to be determined. Recall that semiclassical dynamics is valid far beyond the logtime, but breaks down on algebraic scales before τ_H [6, 37]. In particular, this implies that in the limit of a very weak perturbation, the exponential scale as previously derived places the exponential behaviour beyond τ_H. The semiclassical expression turns out to require modification. Since it is valid on timescales shorter than τ_H, we can expand the exponential to the linear time term. This implies that in this regime the argument of the exponential multiplied by $1 - f$ must be equal to the argument found in the previous subsection. Therefore, to a good approximation, the overlap can be written as

$$\begin{aligned}\overline{A(\epsilon;t)} &\approx f\exp\frac{-\epsilon^2\sigma_v^2t^2}{2\hbar^2}+(1-f)\exp\frac{-\epsilon^2\pi\beta\sigma_v^2\overline{d}t}{2\hbar g(1-f)}\\ &\approx f\exp\frac{-\epsilon^2 gK(E)t^2}{\pi\beta\overline{d}\hbar^3}+(1-f)\exp\frac{-\epsilon^2K(E)t}{\hbar^2(1-f)}\end{aligned}\tag{20}$$

where the first form emphasizes the quantum information (matrix element variance or level slope variance), and the second form emphasizes the classical information (classical action diffusion constant). The transition between the two regimes, Gaussian and exponential, as the strength of the perturbation is increased, is given by the fraction f remaining in the diagonal overlap factor. The exponential decay rate has been modified by the introduction of $1 - f$ in the denominator which is necessary to offset the reduced normalization. In the limit of large perturbations where $f \to 0$ we recover the previous results for the exponential Fermi golden rule decay, and in the small perturbation limit, $f \to 1$, the Gaussian decay.

3.1. Degenerate perturbation theory within RMT

To derive the leading corrections to f for both time-reversal-invariant and noninvariant cases, the most direct approach would be to use the second-order perturbation theory expression

$$\left|\langle n_{\lambda_1}|n_{\lambda_2}\rangle\right|^2 \approx 1-\epsilon^2\sum_{m\neq n}\frac{|V_{mn}|^2}{[E_n(\lambda_1)-E_m(\lambda_1)]^2}+\cdots\tag{21}$$

where $V_{mn} = \langle m_{\lambda_1}|V|n_{\lambda_1}\rangle$. Within RMT, V is a Gaussian random matrix with the correct symmetries, Gaussian orthogonal ensemble (GOE), unitary ensemble (GUE), etc. We only consider the time-reversal-invariant and time-reversal-noninvariant cases, GOE and GUE, respectively.

Define an energy variable s rescaled by the mean local spacing and a unit variance variable x for the perturbation matrix elements. Then the above sum can be reexpressed as the following double integral

$$f \approx 1-2\Lambda\int_{-\infty}^{\infty}\int_0^{\infty}\frac{x^2}{s'^2}R_2(s')\rho(x;s')\,\mathrm{d}s'\,\mathrm{d}x\tag{22}$$

where $\Lambda = \epsilon^2 v^2 \overline{d}_r^2$ is the transition parameter and v^2 is the local variance of the off-diagonal matrix elements of V [26]. $\overline{d}_r$ and v^2 are defined for the irreducible representations of the system. If there are symmetries, then this quantity must be adjusted accordingly by dividing out these symmetries, e.g. $\overline{d}_r = \overline{d}/g$ and $v^2 = \beta\sigma_v^2/2$. The function $R_2(s)$ is a sequence of δ-functions which determine the spacings, and is known as the two-point density correlation function. Similarly, the function $\rho(x; s)$ is a sequence of δ-functions whose intensities determine the matrix elements. Crudely speaking, taking the ensemble average of the above expression is reduced to inserting the $R_2(s)$ function from the appropriate Gaussian ensemble, and taking $\rho(x; s)$ to be a zero-centred, unit variance, Gaussian probability density independent of s.

It was emphasized by French and co-workers that the transition parameter Λ was quite general [26]. Although it naturally emerged from perturbation theories, its relevance extended to entire symmetry breaking transitions and it is the only scale that can enter those problems; a similar statement applies to parametric statistics as well [38].

It is straightforward to understand that for s near zero, the integral diverges for the GOE, and that this approach must fail to give proper results for f. The two-point correlation function is approximately linear, $R_2(s) \approx \pi^2 s/6$, so the above integral will diverge logarithmically. However, we can apply a degenerate perturbation theory approach developed in [26, 39]; the degenerate theory also gives an extra valid correction for the GUE. By diagonalizing a 2×2 matrix, the difference in eigenenergies is given by [40]

$$\Delta E_n = \frac{1}{2}\sum_m \left\{\sqrt{[E_n(\lambda_1) - E_m(\lambda_1)]^2 + 4\epsilon^2|V_{mn}|^2} - [E_n(\lambda_1) - E_m(\lambda_1)]\right\} \times \operatorname{sign}(E_n(\lambda_1) - E_m(\lambda_1)). \tag{23}$$

Using this result, equation (22) can be reexpressed as

$$f \approx 1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_0^{\infty}\left(1 - \frac{s'}{\sqrt{s'^2 + 4\Lambda x^2}}\right) R_2(s')\exp(-x^2/2)\,\mathrm{d}s'\,\mathrm{d}x. \tag{24}$$

Away from very small s', the degenerate perturbation result is unnecessary, but still gives the exact leading order results as a function of Λ. This integral is rather difficult to evaluate exactly, but the correct leading order results can be extracted by splitting the integral into two regions $[0, s]$ and $[s, \infty]$. First, the eigenstate overlap is written as

$$1 - f \approx \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_0^{s}\left(1 - \frac{s\prime}{\sqrt{s'^2 + 4\Lambda x^2}}\right) R_2(s')\exp(-x^2/2)\,\mathrm{d}s'\,\mathrm{d}x + \int_s^{\infty}\frac{2\Lambda}{s'^2}R_2(s')\,\mathrm{d}s'\right] \tag{25}$$

where the power series for $R_2(s)$ is substituted in the first term, and the exact function is left in the second term. For our purposes, it is sufficient to replace $R_2(s)$ in the first region by $\pi^2 s/6$, which gives the GOE result, or by $\pi^2 s^2/3$, which gives the GUE result. The exact two-point correlation functions are [2]

$$R_2(r) = \begin{cases} 1 - [s(r)]^2 + Js(r)Ds(r) & \text{GOE} \\ 1 - [s(r)]^2 & \text{GUE} \end{cases} \tag{26}$$

where $s(r) = \sin(\pi r)/\pi r$, $Ds(r) = \mathrm{d}s(r)/\mathrm{d}r$ and $Js(r) = \int_0^r s(t)\,\mathrm{d}t - 1/2$. Next, the $[0, s]$ integrals are evaluated exactly, and their asymptotic expansions kept up to (not including) $O(\Lambda^2)$. The other integral is also exactly integrated, and its small-s expansion is made. When properly formulated and the two evaluations summed, all the divergent terms less than $O(\Lambda^2)$ vanish; also the s-dependent terms coming from the $[0, s]$ integration vanish. The

s-independent terms give the desired result. The remaining s-dependent terms arising from the $[s, \infty]$ integral vanish as higher-order terms in the power series of $R_2(s)$ are included in the first integral. In this way, the final result does not depend on the value chosen for s. Carrying out this technique up to but not including $O(\Lambda^2)$, we obtain

$$f \approx \begin{cases} 1 - \frac{\pi^2\Lambda}{6}\left(1 - \gamma - \ln\left|\frac{\pi^2\Lambda}{4}\right|\right) & \text{GOE} \\ 1 - \frac{2\pi^2\Lambda}{3} + \frac{32\pi\Lambda}{9}\sqrt{2\pi\Lambda} & \text{GUE} \end{cases} \tag{27}$$

where γ is the Euler constant. The GOE results and the correction term $(\Lambda^{3/2})$ for the GUE are new, and the leading GUE term is consistent with [34], which also gives the asymptotic results

$$f \sim \begin{cases} \frac{1}{\pi^2\Lambda} & \text{GOE} \\ \frac{1}{\pi^2\Lambda} & \text{GUE.} \end{cases} \tag{28}$$

The overlap ensemble average is expected to be universal and should apply to all systems chaotic enough to expect RMT behaviour. Numerical studies and heuristic arguments suggest that the overlap between perturbed and unperturbed eigenstates is roughly a Lorentzian-like function of the perturbation strength [35, 36]. We look for a dependence of the form

$$f = \frac{1}{1 + h(\Lambda)}. \tag{29}$$

Note that in the previously discussed forms, the expression for f was raised to the power β, $h(\Lambda) \approx a_\beta \Lambda$, and approximate values of the constants a_β were given by a numerical fit. Using perturbation theory Wilkinson and Walker [41] gave an expression for the individual off-diagonal matrix elements which has a compatible form. Nevertheless, the $O(\Lambda \ln \Lambda)$ and $O(\Lambda^{3/2})$ corrections plus the asymptotic expressions indicate difficulties with their interpolation since it gets both limits of small and large Λ incorrectly. Equations (27)–(29) are, of course, insufficient to determine $h(\Lambda)$. However, we suggest the following as perhaps the simplest forms for somewhat improved interpolations that are at least analytically consistent with the limiting regimes of large and small Λ:

$$h(\Lambda) \approx \begin{cases} \frac{\pi^2\Lambda}{6}\left(1 - \gamma - \left(\frac{1+b_0\pi^4\Lambda^2}{1+b_1\pi^2\Lambda+b_0\pi^4\Lambda^2}\right)\ln\left|\frac{\pi^2\Lambda}{4+\exp(5+\gamma)\pi^2\Lambda}\right|\right) & \text{GOE} \\ \frac{2\pi^2\Lambda}{3}\left(\frac{1+\frac{3}{2}c_0\pi^2\Lambda}{1+\frac{16}{3}\sqrt{\frac{2}{\pi}\Lambda}+c_0\pi^2\Lambda}\right) & \text{GUE.} \end{cases} \tag{30}$$

The use of a quotient of two polynomials inserts a general function with some fitting parameters that we determined with GOE and GUE Monte Carlo simulations; see the appendix for the Monte Carlo details. Our best fits are: $b_0 = 0.932, b_1 = 0.618$ and $c_0 = 1.89$. Figure 1 shows the differences between the Monte Carlo results and the new interpolations along with the differences between the Monte Carlo results and those from [35, 36]. Even though the new interpolations have the correct limiting forms, they still appear to have some small, detectable deviations from the exact RMT predictions. Perhaps, the main conclusion is that the previous interpolations do surprisingly well given that they fail to get any of the limiting cases correctly.

Again, semiclassical theory fixes the value of Λ using the arguments presented in the previous section. Thus,

$$\Lambda = \frac{\epsilon^2\beta\sigma_v^2\overline{d}^2}{2g^2} = \frac{\epsilon^2\overline{d}K(E)}{\pi g\hbar} \tag{31}$$

where the last form follows from equation (8). In some simple systems such as the standard map used in the next section, $K(E)$ can be calculated analytically. In fact, the criterium,

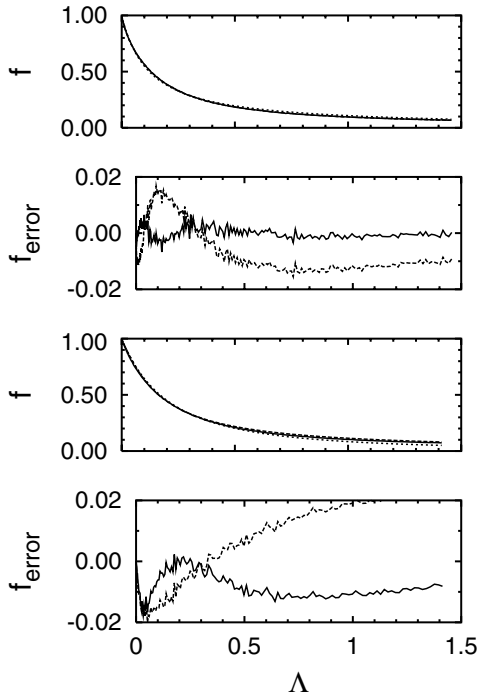

Figure 1. Overlap of eigenstates, f. The two upper and two lower plots are for GOE and GUE statistics, respectively. The solid curves in the upper and lower middle plots are results from RMT calculations for an ensemble of 300 random (300×300) matrices. The dashed curves are the uniform interpolation, while the dotted curves are the Lorentzian ansatz of [35]. The upper middle and lower plots show the difference between RMT and the uniform interpolation (solid) and the difference between RMT and the Lorentzian ansatz (dashed).

equation (18), for observing either Gaussian or exponential behaviour can be written in terms of the transition parameter

$$\pi^2 \Lambda \overset{?}{\gtrless} \frac{1}{\beta}. \tag{32}$$

It is remarkable that the same transition parameter that determines the effects of weak symmetry breaking in RMT, gives precisely the exponential-to-Gaussian crossover for the fidelity.

Finally, the uniform approximation for the fidelity is the square of equation (20) with the results given in equations (29)–(31). It could have turned out that there were three parameters in the functional form of the decay, the relative amount of Gaussian to exponential component (f), and the scales in the arguments of the Gaussian and exponential functions. Interestingly, other than the density of states (or phase-space volume), only one quantity determines all three potential parameters, that being the classical action diffusion constant. There is no other information contained in the decay of the fidelity for a strongly chaotic system.

4. Fidelity in the quantized standard map

We use the standard map, which is a paradigm for chaotic systems, to demonstrate our results for time-reversal-invariant systems. For a kicking strength λ greater than 6 or so the standard

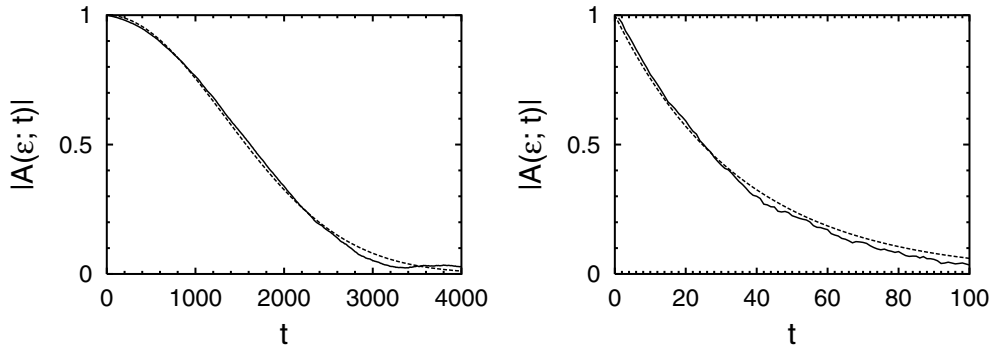

Figure 2. Example of Gaussian and exponential decays. The quantum standard map curves are solid and the theoretical curves are dashed. The left plot is for $\epsilon = 2 \times 10^{-3}$ and the right plot is for $\epsilon = 10^{-4}$. The other parameters are $N = 1000$ and $\lambda = 18$.

map is fully chaotic. The classical map is defined by

$$\begin{aligned} p_{i+1} &= p_i - (\lambda/2\pi)\sin(2\pi q_i) \qquad \text{mod}(1) \\ q_{i+1} &= q_i + p_{i+1} \qquad\qquad\qquad\quad \text{mod}(1). \end{aligned} \tag{33}$$

The quantized propagator with N discrete levels is given by

$$\langle n'|\hat{U}|n\rangle = \frac{1}{\sqrt{\mathrm{i}N}} \exp[\mathrm{i}\pi(n-n')^2/N] \exp\left(\mathrm{i}\frac{kN}{2\pi}\cos[2\pi(n'+a)/N]\right) \tag{34}$$

where $n, n' = 0, \ldots, N-1$ and a is a phase term which we set equal to zero. The effective Planck constant is $h = 1/N$ and the average density of states is $\overline{d} = N/2\pi$.

From the considerations of section 2, examples of nearly pure Gaussian or exponential decay can be found by selecting an appropriate value of ϵ using equation (18). In figure 2, we see that for small $\hbar$ (large N), even without initial state averaging, the theoretical curves for the Gaussian and exponential regimes respectively match the quantum results. The theory curves use the analytic form of the action diffusion constant, which for the standard map is [30]

$$K(E) \approx [1 + 2J_2(\lambda)]/4(2\pi)^4. \tag{35}$$

Also, it turns out that the expressions for the perturbation matrix element variance or level slope variance are different for quantized maps than for continuous dynamical systems since the Floquet eigenangles are scaled differently than energy eigenvalues. Here, the variance of the eigenangle slopes is

$$\sigma_\phi^2 = \sigma_v^2/\hbar^2 = g(4\pi)^2 N K(E) \qquad \text{where} \quad g = 2 \quad \beta = 1. \tag{36}$$

In figure 3, we show an example of the behaviour of f as a function of the transition parameter Λ. Ergodicity allows the ensemble averaging to be replaced by an average over eigenstates. For quantum maps, the transition parameter is defined in terms of the variance of the eigenphases as opposed to the eigenenergies as in continuous systems. Therefore, for the quantized standard map

$$\Lambda = \frac{\epsilon^2\beta\sigma_\phi^2\overline{d}^2}{2g^2} = \frac{2\epsilon^2 N^3 K(E)}{g} \approx \frac{\epsilon^2 N^3}{8(2\pi)^4}[1 + 2J_2(\lambda)]. \tag{37}$$

Using the GOE interpolation expression for f in equation (30) gives excellent agreement with the quantum results.

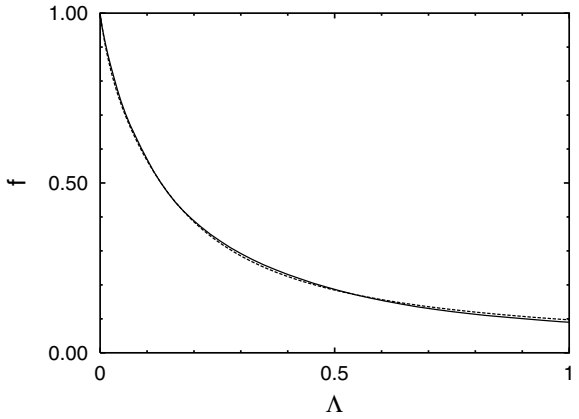

Figure 3. Overlap of eigenstates, f, for the quantum standard map. The solid curve is for the quantum standard map averaged over eigenstates for $N = 1000$ and $\lambda = 18$. The dashed curve is the interpolation formula using RMT and degenerate pertubation theory.

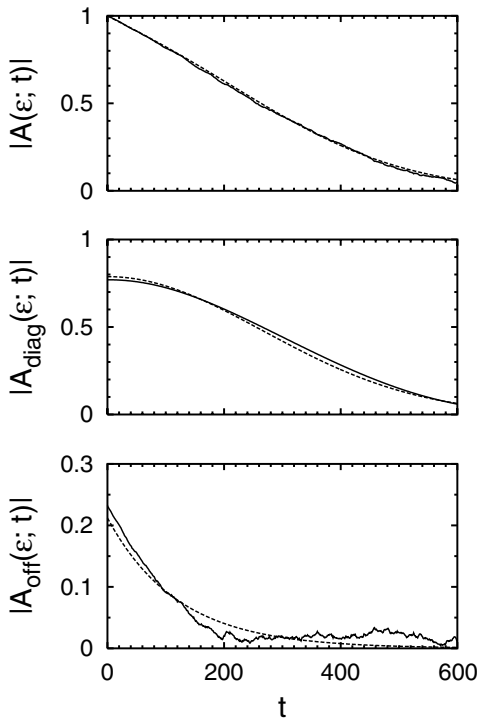

Figure 4. Crossover decay. The upper panel demonstrates the agreement between the quantum standard map (solid curve) and the theoretical results (dashed curve). The middle and lower panels show each piece of the fidelity (diagonal and off-diagonal) separately and the corresponding theoretical results (Gaussian and exponential). The parameters are $N = 1000, \lambda = 18$ and $\epsilon = 5 \times 10^{-4}$.

Finally, we show an example in figure 4 where ϵ has been chosen so that $A(\epsilon; t)$ exhibits both exponential and Gaussian behaviours in different time regimes. The theoretical expression given in equation (20) with equation (35) and the interpolation approximation for f of the

previous section compare well over the full time dependence. It is possible to isolate both the exponential and Gaussian components of $A(\epsilon; t)$, and these are shown in the lower panels. The intercepts of the curves at $t = 0$ show the accuracy of the numerical fit to f, and the decay scales can be seen to be given properly as well.

5. Conclusions

The fidelity is extremely useful for studies of decoherence, reversibility and tomography in several contexts, and has received a great deal of attention recently. It is to be expected that RMT theory would be helpful in understanding its behaviour for strongly chaotic systems. Indeed, we applied RMT to develop a uniform approximation that covers both previously known regimes of Gaussian and exponential decay, and the theoretical predictions describe the behaviour of the quantized standard map extremely well. Although not noted earlier in the text, the range of Λ over which the bare degenerate perturbation theory is accurate is disappointingly narrow. However, it helped to provide a schema for an improved interpolation formula. We further included the semiclassical analysis required to show that only one parameter (other than the density of states/phase-space volume) determined the full behaviour, the classical action diffusion constant. In quantum terms, the sole information is carried by the transition parameter or alternatively by the perturbation spreading width. As a function of this parameter, the uniform approximation gives a continuous family of universal decay curves for all strongly chaotic systems.

Our numerical tests of the time-reversal-invariant uniform approximation with the quantum kicked rotor relied on large dimensionality examples, i.e. $N = 1000$, to show that although we averaged over initial states to generate the theory, the ergodic properties of RMT imply that the results apply individually to nearly any initial state of a strongly chaotic system. For smaller values of N, the fluctuations due to not making an initial state averaging increase and are readily visible. There, averaging would be required to have the same high quality comparison between the theoretical prediction and the quantum results. We would expect a similar quality of results to apply for time-reversal-noninvariant chaotic systems.

The new interpolation formulae given in equation (30) for the overlap of a perturbed and an unperturbed eigenstate are improvements over the simple Lorentzian forms since they contain the correct limiting forms, and are more faithful to the Monte Carlo simulations. Nevertheless, the previous forms work surprisingly well; even where the slope at small perturbation goes to infinity in the GOE results, the previous form never goes too far astray. The overlap f is important in other contexts as well. For example, in mesoscopic physics f is related to the conductance of a quantum dot undergoing a parametric change. For reasons such as this, it would be interesting to know the full analytic transition with perturbation strength between unity and zero.

For integrable, near-integrable and mixed phase-space systems, the behaviour of the fidelity is not universal and depends upon the initial state chosen. Similarly, chaotic systems in the Lyapunov regime (not studied in this paper) could also exhibit initial state dependent phenomena. Such systems are also of interest and contain a variety of different regimes; see the work in [22, 23]. We are currently pursuing several open questions in these dynamical systems.

Acknowledgments

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Appendix. Monte Carlo simulations of f for the GOE and GUE

We use the Hamiltonian

$$H = \sin(\epsilon)H_0 + \cos(\epsilon)H_1 \tag{A.1}$$

in the Monte Carlo simulations. H_0 and H_1 are random (300×300) matrices in either the GOE or GUE symmetry. New random matrices are recalculated for each ϵ to generate independent samples. An ensemble of 300 such matrices is used where the average density of states, $\overline{d}$, and variance of the level velocities, σ_E^2, are averaged over the entire ensemble. Only the middle third of the spectrum is used. The eigenfunctions of H_0 are overlapped with those H and then the absolute value square is taken to obtain f. Λ is given by

$$\Lambda = \begin{cases} \epsilon^2 \sigma_E^2 \overline{d}^2 / 2 & \text{GOE} \\ \epsilon^2 \sigma_E^2 \overline{d}^2 & \text{GUE.} \end{cases} \tag{A.2}$$

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